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Book Review:

Theory and Applications of Stochastic Differential Equations By Z. Schuss
Wiley-Interscience, 321 p., $23.36

The term "stochastic differential equations" occurs in the literature in at least three different meanings: Firstly to denote the class of all differential equations whose coefficients are random numbers or functions. Secondly for that particular subclass in which these random coefficients consist of Gaussian white noise. Such equations are also called Itô equations. Finally the term is sometimes applied to differential equations that are not random themselves but are related to stochastic problems, such as the "stochastic Liouville equation." It is the second definition that is well-nigh universal in the mathematical literature and is also the one referred to in the title of this book.

The presence of white noise gives rise to peculiar mathematical difficulties due to the fact that white noise is not a properly defined stochastic process. These difficulties were recognized by Itô forty years ago and he developed a mathematical scheme in which a meaning could be attached to such equations. The unfamiliar rules of Itô's calculus—or rather his abrogation of the familiar rules of calculus—has led many physicists astray, not to speak of chemists. As most of the mathematical literature on the subject is too opaque to illuminate them, they will welcome a book in which the essentials of this calculus are explained in more familiar language and in close connection with applications. It might even be welcomed by those who agree with me that the Itô calculus is a red herring.

How does one make this herring palatable? The author's strategy is to describe and explain the more formal aspects of the manipulations in the main text, and to relegate his mathematical scruples to special sections. He defines in this manner the stochastic integral according to Itô and also gives the alternative definition of Stratonovich. The result is applied to the interpretation of the Itô equation. One is a bit surprised to read in this context the casual remark that Itô is not suitable for describing physical phenomena, but this remark does not appear to have any consequences. Subsequently the connection with Fokker–Planck–Kolmogorov–Kramers

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is established, so that from now on the problem is reduced to solving partial differential equations.

From here on the interest is focused on these nonstochastic differential equations. They are used to study exit problems and diffusion across barriers. For small diffusion constant a singular perturbation theory for partial differential equations is needed, which is the author's main theme and interest. This is also the part of the book that is the most valuable contribution to the physicist's education, because the present literature abounds with studies on fluctuations near transition points, written by physicists who often are unaware of the existence of perfectly sensible and usable mathematical methods to deal with them. Henceforth they will have no excuse, in particular since the author makes the connection with the real world clear by applying his singular perturbation theory to chemical reaction rates, atomic migration in crystals, and filtering theory.

The book arose from lectures and is written as a textbook. It starts with a rather standard introduction on probability, contains numerous exercises, and ends with a chapter mainly intended to provide some basic knowledge of partial differential equations. Its drawback as a textbook, however, is that it appears to have been written in a hurry, with scant attention to detail and clarity of presentation. It cannot be studied by students without the assistance of a teacher to elucidate the obscurities of the style, warn for inadvertent changes in notation, correct the numerous misprints, and to tell for instance that one does not have to understand the algebra on page 133 since the final formula can be obtained immediately.

One historical remark: Brown was not the first to observe the Brownian motion, Fokker and Planck were anticipated by Rayleigh, the Kramers equation was first written down by O. Klein, and the Ornstein–Uhlenbeck process was derived in a paper by Uhlenbeck and Ornstein. I do not advocate changing the established nomenclature, but it should not be presented as a historical record.

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